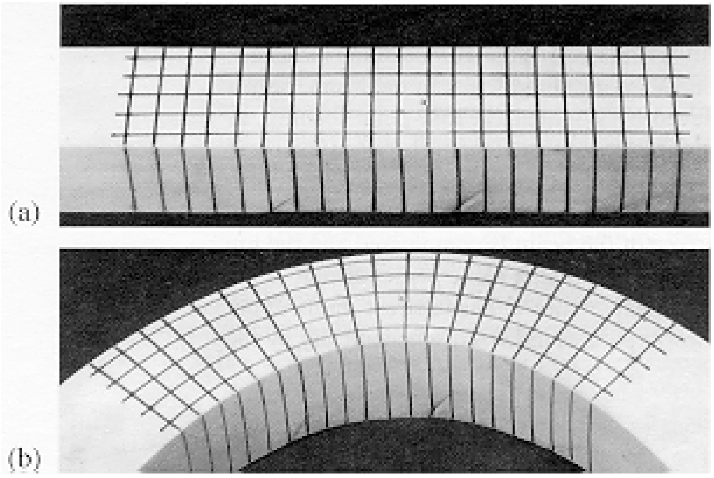
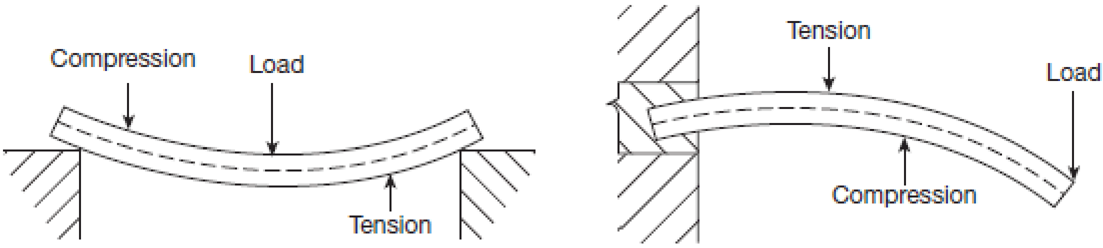
Bending of Beams

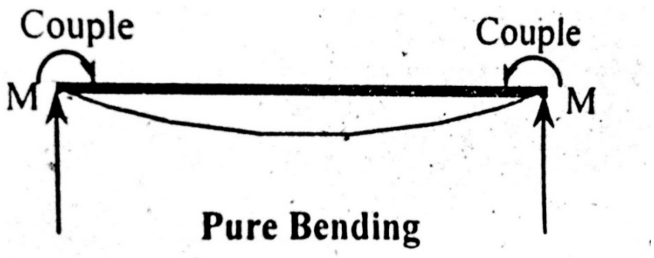
When a 'beam' experiences a bending moment, it will change its shape and internal stresses (forces) will be developed. The photograph illustrates the shape change of elements of a beam in bending. Note that the material is in compression on the inside of the curve and tension on the outside of the curve, and that transverse planes in the material remain parallel to the radius during bending





(a) Simply supported beam (b) Cantilever beam

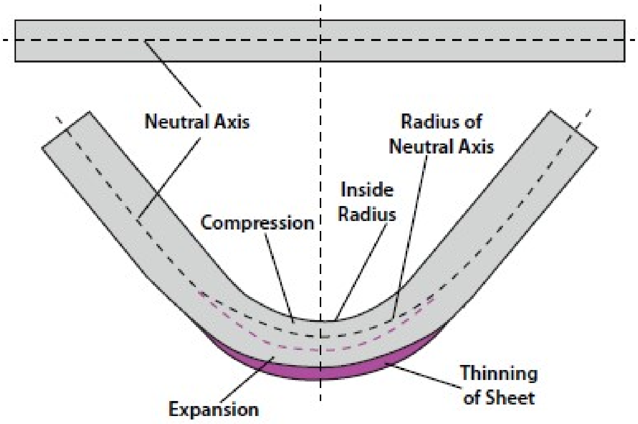
Bending will be called as ***pure bending*** when it occurs solely because of coupling on its end. In that case there is no chance of shear stress in the beam. But the stress that will propagate in the beam as a result will be known as normal stress. Normal stress because it not causing any damages to beam.

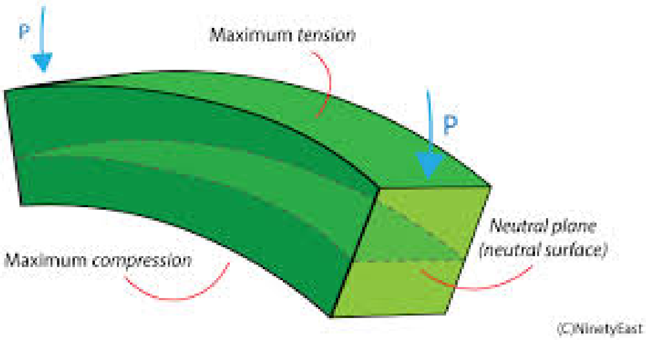


Neutral axis

The neutral axis is an axis in the cross section of a beam (a member resisting bending) or shaft along which there are no longitudinal stresses or strains. If the section is symmetric, isotropic and is not curved before a bend occurs, then the neutral axis is at the geometric centroid. All fibres on one side of the neutral axis are in a state of tension, while those on the opposite side are in compression.

Since the beam is undergoing uniform bending, a plane on the beam remains plane





Bending Equation Derivation

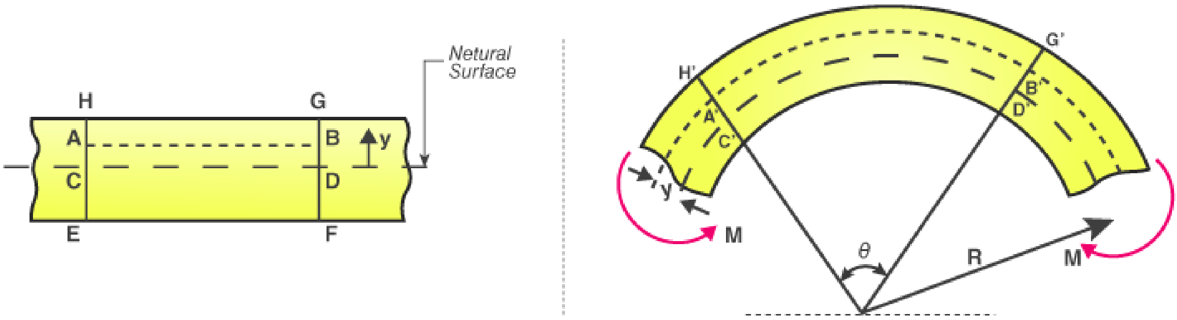
Bending theory is also known as flexure theory is defined as the axial deformation of the beam due to external load that is applied perpendicularly to a longitudinal axis which finds application in applied mechanics.

For a material, flexural strength is defined as the stress that is obtained from the yield just before the flexure test. It represents the highest stress that is experienced within the material at the moment of its yield. σ is used as the symbolic representation of flexural strength.

Assumptions

1. Following are the assumptions made before the derivation of bending equation:
2. The beam used is straight with constant cross section.
3. The beam used is of homogeneous material with a symmetrical longitudinal plane.
4. The plane of symmetry has all the resultant of applied loads.
5. The primary cause of failure is buckling.
6. E remains same for tension and compression.
7. Cross section remains the same before and after bending.

Consider an unstressed beam, which is subjected to a constant bending moment such that the beam bends up to radius R. The top fibres are subjected to tension whereas the bottom fibres are subjected to compression. The locus of points with zero stress is known as neutral axis.



With the help of the above figure, the following are the steps involved in the derivation of the bending equation:

Strain in fibre AB is the ratio of change in length to original length.

Strain in fibre $AB = \frac{A'B' - AB}{AB}$

$\therefore strain = \frac{A'B' - C'D'}{C'D'}$ (as AB = CD and CD = C'D')

CD and C'D' are on the neutral axis and stress is assumed to be zero, therefore strain is also zero on the neutral axis.

$$= \frac{(R + y)\theta - R\theta}{R\theta}$$

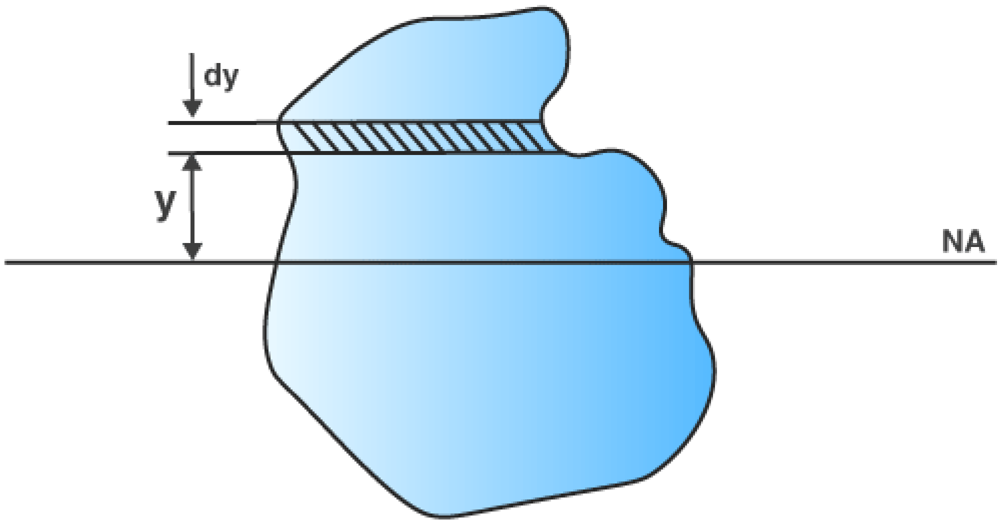
$$= \frac{R\theta - y\theta - R\theta}{R\theta}$$

$$strain = \frac{y}{R}$$

$$\frac{\sigma}{E} = \frac{y}{R}$$

where E is Young's Modulus of Elasticity

$$\frac{\boldsymbol{\sigma}}{\boldsymbol{y}} = \frac{\boldsymbol{E}}{\boldsymbol{R}} \qquad \textbf{eq.1}$$



$$\sigma = \frac{E}{R}y$$

$F = \sigma\delta A = \frac{E}{R}y * \delta A$ (force acting on the strip with area dA)

$F * y = \frac{E}{R}y^2\delta A$ (momentum about neutral axis)

$M = \sum\frac{E}{R}y^2\,\delta A$ (total momentum for entire cross-sectional area)

$$M = \frac{E}{R}\sum y^2\delta A$$

$\sum y^2\delta A$ *is known as second moment of area and is represented as I*

$$M = \frac{E}{R}I$$

$$\boldsymbol{\frac{M}{I} = \frac{E}{R}}$$ **eq.2**

$$\boldsymbol{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$ **From eq.1 and eq.2**

Section Modulus

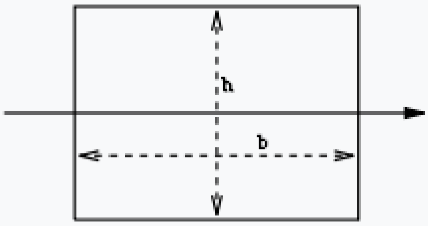
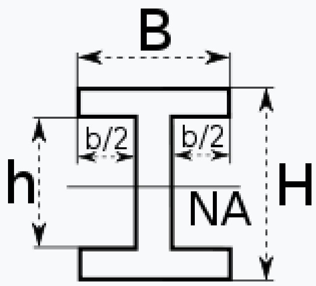
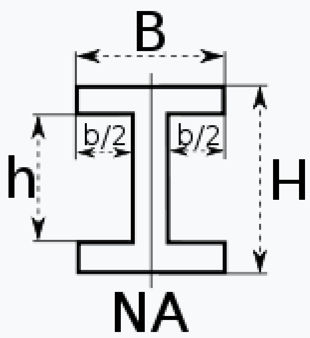
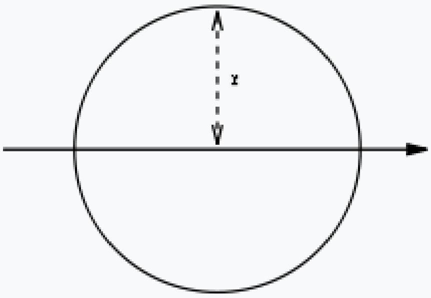
Ratio of Moment of Inertia to the Distance from the Neutral Axis to the Point where the Bending Normal Stress is determined.

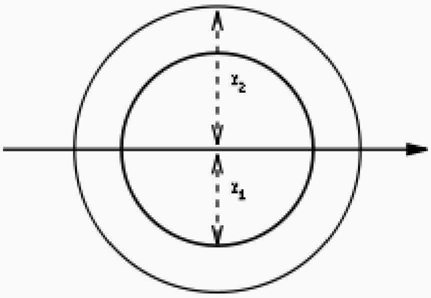
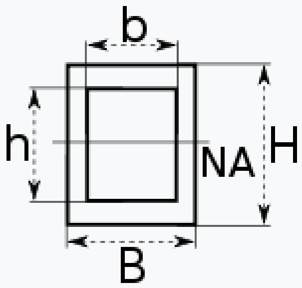
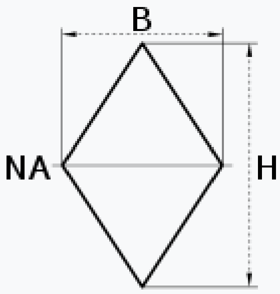
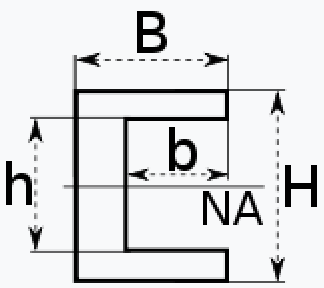
Section modulus (Z) = I/Y

Section modulus is a geometric property for a given cross-section used in the design of beams or flexural members.

Also, **Section modulus** is the **direct Measure of Strength of a beam**. Higher the Section Modulus higher will be the **Resistance to Bending**.

To calculate Z, the distance (y) to the extreme fibres from the centroid (or neutral axis) must be found as that is where the maximum stress could cause failure.

Cross-sectional shape	Figure	Equation	Comment
Rectangle		$S = \frac{bh^2}{6}$	Solid arrow represents neutral axis
doubly symmetric I-section (major axis)		$S = \frac{BH^2}{6} - \frac{bh^3}{6H}$ $Sx = \frac{Ix}{y}$ $y = \frac{H}{2}$	NA indicates neutral axis
doubly symmetric I-section (minor axis)		$Sy = \frac{B^2(H-h)}{6} + \frac{(B-b)^3 h}{6B}$	NA indicates neutral axis
Circle		$S = \frac{\pi d^3}{32}$	Solid arrow represents neutral axis

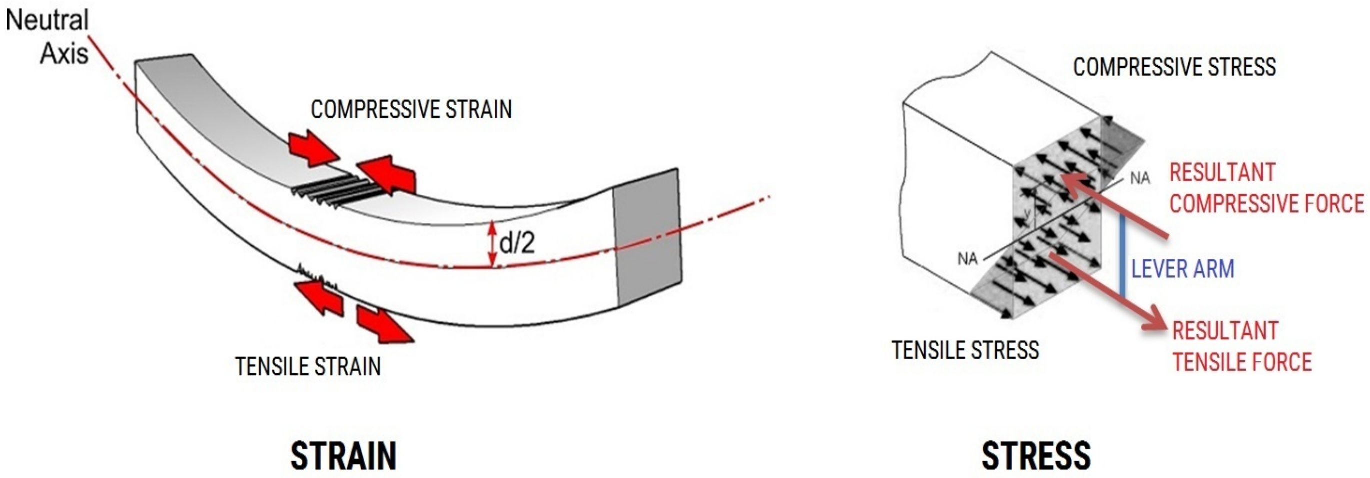
Circular hollow section		$S = \frac{\pi\left(r_2^4 - r_1^4\right)}{4r_2} = \frac{\pi(d_2^4 - d_1^4)}{32d_2}$	Solid arrow represents neutral axis
Rectangular hollow section		$Sx = \frac{BH^2}{6} - \frac{bh^3}{6H}$	NA indicates neutral axis
Diamond		$S = \frac{BH^2}{24}$	NA indicates neutral axis
C-channel		$S = \frac{BH^2}{6} - \frac{bh^3}{6H}$	NA indicates neutral axis

Moment of Resistance

Moment of resistance denotes the resistance offered by the beam to the external moment applied.

In other words, it's the capacity of a beam to withstand the applied moment without failure.

The main playing factor in moment of resistance of a beam is its section modulus (Z)



When a body is strained it tries to resist that by generating internal stresses. Stress is the **resistance** offered by a body to external force.

When a beam bends the concave face of the beam is under compression and the convex face is under tension. These compressive and tensile strains produce compressive and tensile stresses (**resistance**) in the beam respectively. The couple formed by these **resistive forces** is termed as **moment of resistance**.

The sum of the moments of the internal stresses about neutral axis is known as moment of resistance. The algebraic sum of these moments is equal and opposite to the bending moment acting on the section.

Ultimate Moment of Resistance

If we consider the compressive and tensile stress in the beam to be equal to the tensile and compressive strength of the material then the couple formed by them is termed as ultimate moment of resistance or the Ultimate Bending Moment since the beam cannot take bending moment more than that.

Steel plate of width 40 mm and of thickness 12 mm is bent into a circular arc of radius 12 meters. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take E = 2*10^5 N/mm^2.

Solution

Width of plate b = 40 mm

Thickness of plate t =12 mm

Radius of curvature R = 12 m = 12*1000 mm

$$I = \frac{bt^3}{12} = \frac{40 * 12^3}{12} = 5760\ mm^4$$

To determine maximum bending stress

$$\frac{f}{y} = \frac{E}{R}$$

$$f = \frac{E}{R} * y$$

$$y_{max} = \frac{t}{2} = \frac{12}{2} = 6\ mm$$

$$f_{max} = \frac{2 * 10^5}{12 * 10^3} * 6 = 100\ N/mm^2$$

To determine Bending moment

$$\frac{M}{I} = \frac{E}{R}$$

$$M = \frac{E}{R} * I$$

$$M = \frac{2 * 10^5}{12 * 10^3} * 5760 = 96000\ N\ mm$$

To what is radius of curvature the steel beam should bend so that stress should not exceed 500 N/mm^2. The beam is of symmetrical section having 160 mm depth. E = 2*10^5 N/mm^2.

Solution

fmax = 500 N/mm^2

Depth of beam, d = 160 mm

And R = ?

The bending stress will be maximum at extreme fibre

$$y_{max} = \frac{160}{2} = 80mm$$

$$\frac{f}{y} = \frac{E}{R}$$

$$R = \frac{E * y}{f}$$

$$R = \frac{2 * 10^5 * 80}{500} = 32000\ mm = 32\ m$$

Compare the strength of solid and hollow circular beams. The two beams are made of some material and are of same weight.

Solution

Let diameter of solid beam = d

Outer diameter of hollow beam = d_1

Inner diameter of hollow beam = d_2

The density of beam = ρ

Weight per unit length of solid beam

$$W_s = \frac{\pi}{4} * d^2 * \rho * g$$

Weight per unit length of hollow beam

$$W_h = \frac{\pi}{4} * (d_1^2 - d_2^2) * \rho * g$$

$$given\ W_s = W_h$$

$$\frac{\pi}{4} * d^2 * \rho * g = \frac{\pi}{4} * (d_1^2 - d_2^2) * \rho * g$$

$$\boldsymbol{d^2 = \left(d_1^2 - d_2^2\right)} \quad \boldsymbol{eq1}$$

The moment of inertia of solid section

$$I_S = \frac{\pi}{64} d^4$$

For solid section the distance of extreme fibre

$$y_S = \frac{d}{2}$$

Section modulus for solid section

$$Z_s = \frac{I_S}{y_S} = \frac{\frac{\pi}{64} d^4}{\frac{d}{2}} = \frac{\pi d^3}{32}$$

The moment of inertia of hollow section

$$I_H = \frac{\pi}{64}(d_1^4 - d_2^4)$$

For solid section the distance of extreme fibre

$$y_H = \frac{d_1}{2}$$

Section modulus for solid section

$$Z_H = \frac{I_H}{y_H} = \frac{\frac{\pi}{64}(d_1^4 - d_2^4)}{\frac{d_1}{2}} = \frac{\pi}{32} * \frac{(d_1^4 - d_2^4)}{d_1}$$

The ratio of strength is given as

$$\frac{Z_H}{Z_s} = \frac{\frac{\pi}{32} * \frac{(d_1^4 - d_2^4)}{d_1}}{\frac{\pi d^3}{32}}$$

$$\frac{Z_H}{Z_s} = \frac{(d_1^4 - d_2^4)}{d_1 * d^3}$$

$$\frac{Z_H}{Z_s} = \frac{(d_1^2 - d_2^2) * (d_1^2 + d_2^2)}{d_1 * d^3}$$

$$\boldsymbol{from\ eq1} \quad \boldsymbol{d^2 = \left(d_1^2 - d_2^2\right)}$$

$$\frac{Z_H}{Z_s} = \frac{d^2 * (d_1^2 + d_2^2)}{d_1 * d^3} = \frac{d_1^2 + d_2^2}{d_1 * d}$$

$$\boldsymbol{from\ eq\ 1} \quad \boldsymbol{d_2^2 = d_1^2 - d^2}$$

$$\frac{Z_H}{Z_s} = \frac{d_1^2 + d_1^2 - d^2}{d_1 * d} = \frac{2d_1^2 - d^2}{d_1 * d}$$

$$\frac{Z_H}{Z_s} = \frac{d_1}{d}\left(2 - \frac{d^2}{d_1^2}\right)$$